$SU(N_c)$ gauge theories for all N_c in 3 and 4 dimensions.

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Abstract

We compare the mass spectra and string tensions of $SU(2)$, $SU(3)$ and $SU(4)$ gauge theories in 2+1 dimensions. We find that the ratios of masses are, to a first approximation, independent of N_c and that the remaining dependence can be accurately reproduced by a simple $O(1/N_c^2)$ correction. This provides us with a prediction of these mass ratios for all $SU(N_c)$ theories in 2+1 dimensions and demonstrates that these theories are 'close' to $N_c = \infty$ for $N_c \geq 2$. We also find that, when expressed in units of the dynamical length scale of the theory, the dimensionful coupling g^2 is proportional to $1/N_c$ at large N_c. We confirm that these theories are indeed confining in the limit $N_c \to \infty$. We describe preliminary calculations in 3+1 dimensions which indicate that the same will be true there.

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1 Introduction.

The proposal to consider $SU(N_c)$ gauge theories as perturbations in powers of $1/N_c$ around $N_c = \infty$ is an old one [1]. If one assumes confinement for all N_c, then the phenomenology of the $SU(\infty)$ quark-gluon theory is strikingly similar to that of (the non-baryonic sector of) QCD [1, 2]. This makes it conceivable that the physically interesting $SU(3)$ theory could be largely understood by solving the much simpler $SU(\infty)$ theory [3]. The fact that the lattice $SU(\infty)$ theory can be re-expressed as a single plaquette theory [4], has provided the basis of a number of interesting computational explorations (for a review see [5]). Unfortunately this latter scheme makes no statement about the size of even the leading corrections to the $N_c = \infty$ limit, and so gives us no clue as to how close $N_c = 3$ is to $N_c = \infty$.

In this paper we calculate the properties of $SU(N_c)$ gauge theories for several values of N_c and explicitly determine how the physics varies as N_c increases. We note that there are good reasons for believing that the inclusion of quarks would not alter any of our conclusions (except in some obvious ways). We consider both 2+1 and 3+1 dimensions. The former calculations are much the more precise and it is there that we will be able to make some firm statements. In $D = 3 + 1$ our conclusions will be similar but more tentative.

While one might naively expect that the $D = 2 + 1$ and $D = 3 + 1$ gauge theories would be so different as to make a unified treatment misleading, this is not in fact so. Theoretically the $D = 2 + 1$ theory shares with its $D = 3 + 1$ homologues three central properties. Firstly, at short distances, the dimensionless coupling becomes weak in both cases. In the $D = 2+1$ theory the coupling, g^2 has dimensions of mass so that the effective dimensionless expansion parameter on a scale a will be ag^2 which vanishes linearly with distance (the theory is super-renormalisable). In the $D = 3 + 1$ theory the coupling vanishes logarithmically with distance (asymptotic freedom). Secondly, at large distances both theories appear to be confining, with a non-perturbative linear potential between fundamental sources. Thirdly it is the value of the coupling that sets the overall mass scale in both cases. In $D = 2+1$ this arises directly because g^2 has dimensions of mass. In $D = 3 + 1$ it does so through the phenomenon of dimensional transmutation: the classical scale invariance is anomalous, the coupling runs and this introduces a mass scale through the rate at which it runs (i.e. the $\Lambda_{\overline{MS}}$ parameter). In addition to these general theoretical similarities, the calculated spectra also show some striking similarities. All this motivates us to believe that a unified treatment makes sense.

The $D = 2+1$ analysis is based on our calculations over the last few years of the properties

of $SU(N_c)$ gauge theories with $N_c = 2, 3$ and 4. In $D = 3 + 1$ what we have done is to perform some $SU(4)$ calculations to supplement what is known about $SU(2)$ and $SU(3)$. Our strategy is the very simple one of directly calculating the mass spectra of these theories and seeing whether they are approximately independent of N_c. The calculations are performed through the Monte Carlo simulation of the corresponding lattice theories. In the $D = 2+1$ case the calculations are very accurate and we are able to extrapolate our mass ratios to the continuum limit prior to the comparison. In the 3+1 dimensional case our $SU(4)$ calculations are not good enough for that, and our comparisons with $SU(2)$ and $SU(3)$ are correspondingly less precise.

This study was originally motivated by the observation that the $C = +$ sector of the light mass spectrum turned out to be quite similar in the $D = 2 + 1$ $SU(2)$ [6] and $SU(3)$ [7] theories. (This also appears to be the case in $D = 3 + 1$, although there the comparison is weakened by the much larger errors.) If the reason for this is that both are close to the $N_c = \infty$ limit, then this provides an economical understanding of the spectra of $SU(N_c)$ gauge theories for all N_c, i.e. there is a common spectrum with small corrections.

A second reason for studying $N_c \to \infty$ is that models and theoretical approaches are usually simpler in that limit. For example, the flux tube model of glueballs [8, 9] would naively appear to be identical for $N_c > 2$. However, because the model does not incorporate the effects of glueball decay, it should in fact be tested against the $N_c \to \infty$ spectrum since it is only in that limit that there are no decays. A second example is provided by the recent progress in calculating the large N_c mass spectrum using light-front quantisation techniques [10].

The calculations in this paper have all been performed with the standard plaquette action using standard Monte Carlo techniques. The lattice spacing, a, is varied by changing the dimensionless bare inverse coupling, β, which appears in the lattice action. In the pure gauge theory the states are necessarily composed entirely of gluons and we shall therefore refer to them as 'glueballs'. Some of the $SU(2)$ results have been published [6, 11] and a paper on the other work is in preparation. A brief summary of the work in this paper has been presented elsewhere [12].

2 2+1 dimensions.

We begin with the string tension, σ, since it turns out to be the most accurate physical quantity in our calculations. We use smeared Polyakov loops [13], to obtain $a^2\sigma$ for several values of the lattice spacing a. We then extrapolate the lattice results, using the

asymptotic relation $\beta = 4/ag^2$, to obtain the continuum string tension in units of g^2:

$$\frac{\surd\sigma}{g^2} = \lim_{\beta\to\infty} \frac{\beta}{4} a\surd\sigma \tag{1}$$

The results for $SU(2)$ [11], $SU(3)$ and $SU(4)$ [7], in $D = 2 + 1$ are shown in Table 1. We immediately see that there is an approximate linear rise with N_c and we find that we can obtain a good fit with

$$\frac{\surd\sigma}{g^2} = 0.1974(12)N_c - \frac{0.120(8)}{N_c}. \tag{2}$$

We obtain a similar behaviour with the light glueball masses (see below).

Some observations.

• For large N_c, eqn(2) tells us that $\surd\sigma \propto g^2 N_c$. That is to say, the overall mass scale of the theory, call it μ, is proportional to $g^2 N_c$. In other words, in units of the mass scale of the theory

$$g^2 \propto \frac{\mu}{N_c}. \tag{3}$$

While this coincides with the usual expectation based on an analysis of Feynman diagrams, we note that here the argument is fully non-perturbative.

• The string tension is non-zero for all N_c and, in particular, for $N_c \to \infty$ (when expressed in units of $g^2 N_c$ or the lightest glueball masses - see below). This confirms the basic assumption that needs to be made in 4 dimensions in order to extract the usual phenomenology of the large-N_c theory [3, 2].

• In the pure gauge sector one expects (again from an analysis of Feynman diagrams) [3] that the first correction to the large-N_c limit will be $O(1/N_c^2)$ relative to the leading term. The fit in eqn(2) is indeed of this form. We note that if we try a fit with a $O(1/N_c)$ correction instead (which would be appropriate if we had quarks) then we obtain an unacceptably poor χ^2 (corresponding to a confidence level of only $\sim 2\%$ in contrast to the $\sim 45\%$ we obtain for the quadratic correction). We may regard this as providing some non-perturbative support for the diagram-based expectation.

• The coefficient of the correction term is comparable to that of the leading term, suggesting an expansion in powers of $1/N_c$ that is rapidly convergent. Indeed one has to go to $N_c = 1$ before the correction term becomes comparable to the leading term. While the $SU(1)$ theory is completely trivial, we note that the $U(1)$ theory has a zero string tension (in the sense that $\surd\sigma/g^2 = 0$ in the continuum limit).

In addition to the string tension we have calculated part of the mass spectrum. In particular we have calculated the masses of the lightest particles with J^{PC} quantum numbers $J = 0, 1, 2,\ P = \pm$ and $C = \pm$. In some cases we have calculated some of the excited

states for given J^{PC}. Note that since we are in $D = 2 + 1$, states of opposite parity are degenerate as long as $J \neq 0$. This degeneracy is broken by lattice spacing and finite volume corrections. We will present our results separately for the $P = +$ and $P = -$ states so as to provide an explicit check on the presence of any such unwanted corrections.

We begin with the $C = +$ spectrum since the $SU(2)$ spectrum does not contain $C = -$ states. In this case we have masses for three values of N_c, and so can check how good is a fit of the kind in eqn(2). In Fig. 1. we plot the ratio m_G/g^2N_c against $1/N_c^2$ for a selection of the lightest states, G. On this plot a fit of the form in eqn(2) will be a straight line and we show the best such fits. As we can see, the data is consistent with such a $1/N_c^2$ correction being dominant for $N_c \geq 2$. However what is really striking is the lack of *any* apparent N_c dependence for the lightest 0^{++} and 2^{++} states.

In Table 2 we present the results of fitting the $C = +$ states to the form

$$\frac{m_G}{g^2N_c} = R_\infty + \frac{R_{slope}}{N_c^2}. \tag{4}$$

where $R_\infty = \left.\frac{m_G}{g^2N_c}\right/_{N_c=\infty}$. (Note that the errors on the slope and intercept are highly correlated.) For each state we show the confidence level of the fit. These are acceptable suggesting once again that for $N_c \geq 2$ a moderately sized correction of the form $\sim 1/N_c^2$ is all that is needed. Note that since the variation with N_c is small, the exact form of the correction used will not have a large impact on the extrapolation to $N_c = \infty$ (except in estimating the errors).

These calculations confirm our earlier claim that the physical mass scale at large N_c is g^2N_c. So if we consider ratios of m_G to $\sqrt{\sigma}$ (as was explicitly done in [12]) we will find that they have finite non-zero limits as $N_c \to \infty$: that is to say, the large-N_c theory possesses linear confinement.

For the $C = -$ states we only have masses for 2 values of N_c and we cannot therefore check whether a fit of the form in eqn(4) is statistically favoured or not. However given that such a fit has proved accurate for the $C = +$ masses and for the string tension down to $N_c = 2$ it seems entirely resonable to assume that it will be appropriate for $N_c \geq 3$ for the $C = -$ masses. Assuming this we obtain the results shown in Table 3 for the $N_c = \infty$ limit and for the coefficient of the first correction. The 'lever arm' on this extrapolation is, of course, shorter than for the $C = +$ states and that leads to correspondingly larger errors.

The results in the Tables provide us not only with values for the various mass ratios in the limit $N_c \to \infty$ but also, when inserted into eqn(2), predictions for *all* values of N_c.

Finally we remark that we have also calculated the deconfining temperature, T_c, for $SU(2)$

[14] and for $SU(3)$ [7]. Extrapolating as in eqn(4), we find

$$\frac{T_c}{g^2 N_c} = 0.1745(52) + \frac{0.079(23)}{N_c^2}. \tag{5}$$

Of course, extrapolating from $N_c = 2, 3$ is less reliable than extrapolating from $N_c = 3, 4$.

3 3+1 dimensions.

Our knowledge of 4 dimensional gauge theories is much less precise. As far as continuum properties are concerned, quantities that are known with reasonable accuracy include the string tension, the lightest scalar and tensor glueballs, the deconfining temperature and the topological susceptibility. As in 3 dimensions, the $SU(2)$ and $SU(3)$ values are within $\sim 20\%$ of each other, which encourages us to investigate the $SU(4)$ theory so as to see whether we are indeed 'close' to $N_c = \infty$. Of course $SU(4)$ calculations are much slower in $D = 3 + 1$ and the results we present here are of a preliminary nature.

We use the standard plaquette action, and so our first potential hurdle is the presence of the well-known bulk transition that occurs as we increase the inverse bare coupling, $\beta \equiv 2N_c/g^2$, from strong towards weak coupling. To locate this transition we performed a scan on a 10^4 lattice and found that it occurred at $\beta = 10.4 \pm 0.1$. This corresponds to a rather large value of the lattice spacing, a, and so does not lie in the range of couplings within which we shall be working, i.e. β =10.7,10.9 and 11.1.

Our calculation consists of 4000,6000 and 3000 sweeps on 10^4,12^4 and 16^4 lattices at β =10.7,10.9 and 11.1 respectively. Every fifth sweep we calculated correlations of (smeared) gluonic loops and from these we extracted the string tension and the masses of the lightest 0^{++} and 2^{++} particles, using standard techniques [13]. These are presented in Table 4. We also calculated the topological susceptibility, $a^4\chi_t \equiv < Q^2 > /L^4$, where L^4 is the number of lattice sites and Q is the total topological charge. (Note that in $D = 2+1$ there is no such charge.) The charge Q was obtained using a standard cooling method, just as in $SU(3)$ [15]. The calculations were performed every 50 sweeps. Overall this corresponds to rather small statistics and the errors are therefore unlikely to be very reliable.

We see from Table 4 that the most accurate physical quantity in our calculations is the string tension, σ. Can we learn from it how g^2 varies with N_c, just as we did in $D = 2+1$? We focus on a simple aspect of this question: if we compare different $SU(N_c)$ theories at a value of a which is the same in physical units, i.e. for which $a\sqrt{\sigma}$ is the same, does the bare coupling vary as $1/N_c$, i.e. does $\beta \equiv 2N_c/g^2 \propto N_c^2$? We perform this comparison for $\beta_4 = 10.9, 11.1$. (For convenience we shall label β by the value of N_c, i.e. we write it as

β_{N_c}.) To find the corresponding values of β in $SU(2)$ and $SU(3)$ we simply interpolate between the values provided in (for example) [13]. Doing so we find that the values of β corresponding to $\beta_4 = 10.9, 11.1$ are $\beta_3 \simeq 5.972(18), 6.071(24)$ and $\beta_2 \simeq 2.442(9), 2.485(11)$ respectively. If we simply scale β_4 by N_c^2 then what we would have expected to obtain is $\beta_3 \simeq 6.131, 6.244$ and $\beta_2 \simeq 2.725, 2.775$ respectively. Superficially the numbers look to be in the right ballpark, but in fact the agreement is poor. For example $\beta_2 = 2.725$ and $\beta_2 = 2.442$ correspond to values of $a\sqrt{\sigma}$ that differ by about a factor of 3.

This disagreement should not, however, be taken too seriously, since it is well-known that the lattice bare coupling is a very poor perturbative expansion parameter. It is known that one can get a much better expansion parameter if one uses instead the mean-field improved coupling, g_I^2, obtained from g^2 by dividing it by the average plaquette, $< \frac{1}{N_c} TrU_p >$ [16]. Defining $\beta_{N_c}^I \equiv 2N_c/g_I^2(a)$ we find that $\beta_4 = 10.9, 11.1$ correspond to $\beta_4^I = 6.215, 6.474$ respectively. Scaling β_4^I by N_c^2 we would expect the equivalent $SU(3)$ and $SU(2)$ couplings to be given by $\beta_3^I = 3.496, 3.642$ and $\beta_2^I = 1.554, 1.619$. What we actually find is that the equivalent couplings are $\beta_3^I \simeq 3.527(22), 3.649(28)$ and $\beta_2^I = 1.561(10), 1.613(12)$. The agreement is now excellent. That is to say, if the $SU(N_c)$ mean-field improved bare-coupling is defined on a length scale that is related to the physical length scale ($\sqrt{\sigma}$) by some constant factor, then it varies as $g^2 \propto 1/N_c$. This is, of course, the usual diagram-based expectation.

In Fig. 2 we plot the scalar and tensor glueball masses, in units of $\sqrt{\sigma}$, as a function of N_c. For $N_c = 2, 3$ we have used the continuum values. For $N_c = 4$ the calculations are not precise enough to permit an extrapolation to the continuum limit and so we simply present the values that we obtained at $\beta = 10.9$ and 11.1. (We do not use the $\beta = 10.7$ values since they have large errors and there is the danger that the scalar mass may be reduced by its proximity to the critical point at the end of the bulk transition line.) Although the $N_c = 4$ errors are quite large, it certainly seems that there is little variation with N_c for $N_c \geq 2$ and any dependence appears to be consistent with being given by a simple $1/N_c^2$ correction. The fact that these mass ratios appear to have finite non-zero limits, implies that the large-N_c theory is confining.

As mentioned earlier we have also calculated the topological susceptibility. In Fig. 3 we plot the dimensionless ratio $\chi_t^{1/4}/\sqrt{\sigma}$ as a function of N_c. Once again the N_c=2 and 3 values are continuum extrapolations of lattice values [13, 15], while in the case of $SU(4)$ we simply display the lattice values obtained at β=10.9 and 11.1. We remark that for $SU(4)$ one expects, semiclassically, very few small instantons and this is confirmed in our cooling calculations. This has the advantage that the lattice ambiguities that arise when instantons are not much larger than a are reduced as compared to $SU(3)$, and

dramatically reduced as compared to $SU(2)$. This implies that the interesting large-N_c physics of topology (and the related meson physics) should be straightforward to study.

4 Conclusions.

We have calculated the mass spectra and string tensions of gauge theories with $N_c = 2, 3, 4$ in 3 dimensions. We have found that there is only a small variation with N_c and this can be accurately described by a modest $O(1/N_c^2)$ correction. That is to say, such theories are close to their $N_c = \infty$ limit for all values of $N_c \geq 2$. We find that the large-N_c theory is confining and that $g^2 \propto 1/N_c$ when expressed in physical units. This confirms, in a fully non-perturbative way, expectations arrived at from analyses of Feynman diagrams. It simultaneously provides a unified understanding of all our $SU(N_c)$ theories in terms of just the one theory, $SU(\infty)$, with modest corrections to it. In practical terms this means that, from the parameters in our Tables, we know the corresponding masses for *all* values of N_c.

Our calculations in 4 dimensions, while quite preliminary, suggest that the situation is the same there as in 3 dimensions.

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References

[1] G. 't Hooft, Nucl. Phys. B72 (1974) 461.

[2] E. Witten, Nucl. Phys. B160 (1979) 57.

[3] S. Coleman, 1979 Erice Lectures.

[4] T. Eguchi and H. Kawai, Phys. Rev. Lett. 48 (1982) 1063.

[5] S.R. Das, Rev. Mod. Phys. 59(1987)235.

[6] M. Teper, Phys. Lett. B289 (1992) 115.

[7] M. Teper, in preparation.

[8] N. Isgur and J. Paton, Phys. Rev. D31 (1985) 2910.

[9] T. Moretto and M. Teper, heplat-9312035.

[10] F. Antonuccio and S. Dalley, Nucl. Phys. B461 (1996) 275.

[11] M. Teper, Phys. Lett. B311 (1993) 223.

[12] M. Teper, to appear in the Proceedings of Lattice '96.

[13] C. Michael and M. Teper, Nucl. Phys. B305 (1988) 453; B314 (1989) 347.

[14] M. Teper, Phys. Lett. B313 (1993) 417.

[15] M. Teper, Phys. Lett. B202 (1988) 553.

[16] G. Parisi, in *High Energy Physics* - 1980(AIP 1981); G. Lepage and P. Mackenzie, Phys. Rev. D48 (1993) 2250.

N_c	$\sqrt{\sigma}/g^2$
2	0.3350 (15)
3	0.5530 (20)
4	0.7564 (45)

Table 1: The $D = 2 + 1$ $SU(N_c)$ confining string tension.

β	R_∞	R_{slope}	CL(%)
0^{++}	0.805 (13)	-0.06 (8)	90
$0^{++\star}$	1.245 (27)	-0.41 (14)	85
0^{-+}	1.788 (88)	-0.48 (56)	40
2^{++}	1.333 (29)	-0.08 (18)	45
2^{-+}	1.340 (40)	-0.01 (24)	12
1^{++}	1.946 (75)	-0.59 (47)	95
1^{-+}	1.919 (115)	-0.18 (75)	30

Table 2: States with $C = +$ in $D = 2 + 1$: $R_\infty \equiv \lim_{N_c \to \infty} \frac{m_G}{g^2 N_c}$ and R_{slope} is the coefficient of the $1/N_c^2$ correction in eqn(4).

G	R_∞	R_{slope}
0^{--}	1.18 (6)	0.1 (0.6)
$0^{--\star}$	1.47 (10)	0.3 (1.1)
0^{+-}	1.98 (28)	-0.4 (2.7)
2^{--}	1.52 (14)	0.9 (1.4)
2^{+-}	1.58 (13)	-0.4 (1.3)
1^{--}	1.85 (15)	-0.3 (1.5)
1^{+-}	1.78 (23)	1.3 (2.3)

Table 3: As in Table 2 but for states with $C = -$.

β	$a\sqrt{\sigma}$	$am_{0^{++}}$	$am_{2^{++}}$
10.7	0.296 (14)	0.98 (17)	1.78 (34)
10.9	0.229 (7)	0.77 (8)	1.20 (10)
11.1	0.196 (7)	0.78 (6)	1.08 (10)

Table 4: $SU(4)$ in 4 dimensions; masses calculated at the values of β shown.

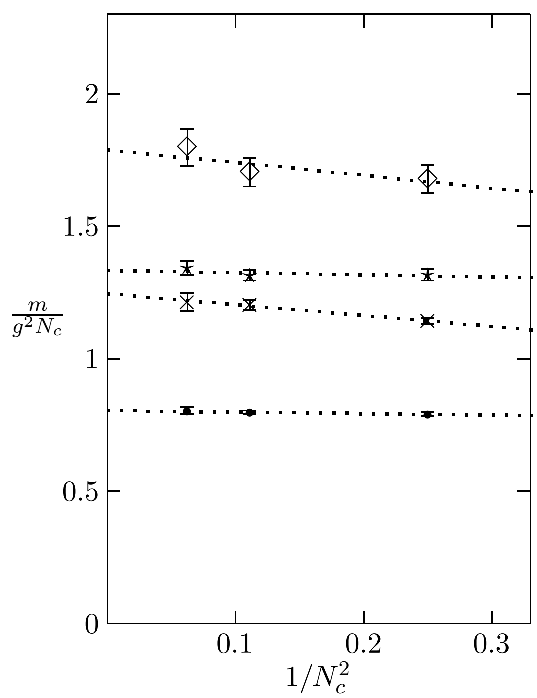

Figure 1: Some continuum glueball masses, in $D = 3$, for 2,3,4 colours: $0^{++}(\bullet)$, $0^{++*}(\times)$, $2^{++}(\star)$, $0^{-+}(\diamond)$ and linear fits.

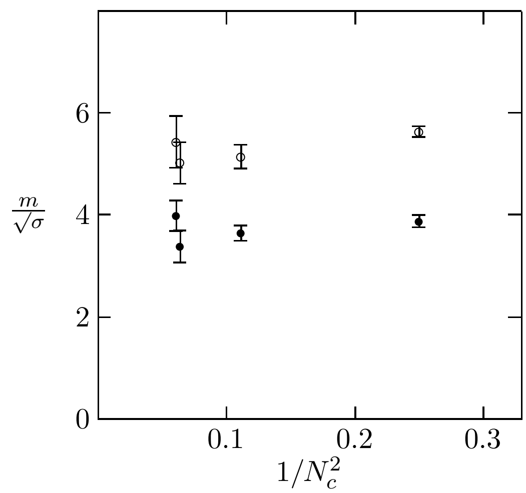

Figure 2: Lightest scalar (•) and tensor (○) glueball masses in $D = 4$. Continuum values for $N_c = 2, 3$ and lattice values ($\beta = 10.9$ and $\beta = 11.1$) for $N_c = 4$.

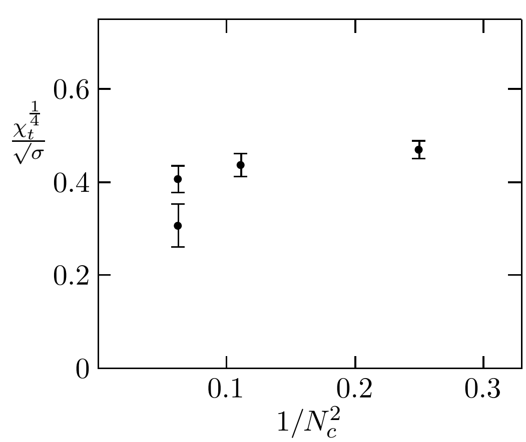

Figure 3: The topological susceptibility: continuum values for $N_c = 2, 3$ and lattice values ($\beta = 10.9$ and $\beta = 11.1$) for $N_c = 4$.